

FRACTAL ANALYSIS OF YIELD MAPS

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ABSTRACT. *Agricultural crop yield data usually are highly nonlinear and complex. Basic mathematical and statistical techniques are sometimes insufficient to describe the nature, trend, or cause of variations in yield. This article investigates fractal analysis of agricultural yield maps and describes a method of applying fractal analysis to multiple years of yield data. It also shows how patterns of yield variation can be described by fractal geometry. Crop yield was measured and mapped for an agricultural field for five consecutive years. In order to obtain a sufficiently dense set of points necessary for valid fractal analysis, a method was proposed to transfer the data points from \mathbb{R}^3 to \mathbb{R}^2 . Analysis of the resulting data set revealed multifractality of the yield variations. It was shown that multifractal measures such as the Rényi spectrum can be used to quantify and compare global and local yield variations.*

Keywords. *Fractal analysis, Multifractals, Precision agriculture, Spatial variability, Yield map.*

Analyzing and interpreting yield data is one of the main steps in successful application of precision agriculture. Assessment and interpretation of yield data can be very difficult because sometimes there are not enough data on the soil or plant factors that cause yield variability. Most of the common attempts to characterize in-field spatial yield variability have used basic statistical measures or geostatistical methods (Eghball et al., 1999; Blackmore et al., 2003). Because in-field variations are highly non-linear and complex, these methods cannot fully describe the variation patterns.

Maps of crop yield variability have been studied more than any other type of variability map because crop yield maps show the final outcome of management decisions and indicate overall field variability better than other variability maps. Spatial variability in crop yield is the result of interactions among many factors, including topography of the terrain, soil physical and chemical properties, and soil water content. These factors generate variations at nested scales that may result in self-similar patterns of variation (Green and Erskine, 2004). If so, these variation patterns can be described by fractal geometry (Peitgen et al., 2004) and quantified by fractal dimensions or fractal dimension spectra. Moreover, using methods such as joint multifractal analysis, the variability of crop yield can be related to other spatial variables of interest (Meneveau et al., 1990). Knowledge of

scale-dependent variation can also be useful when generalizing analysis results or management decisions from a smaller to a larger scale or vice versa (Bekele et al., 2005).

A few studies have used fractal analysis to characterize temporal and spatial variability in an agricultural field. Perfect and Blevins (1997) used fractal analysis of soil structure to compare different tillage practices. Evaluation of the effect of variable-rate fertilizer application strategies has also been completed using fractal analysis of soil nitrate (Eghball et al., 1999). Fractal analysis of temporal and spatial variability in yield data has shown that temporal (year-to-year) variability in crop yield might be more significant than any spatial variability (Eghball and Varvel, 1997). Multifractal and joint multifractal analysis methods have been used to investigate the variability of crop yield with terrain slope (Kravchenko et al., 2000). Furthermore, fractal dimensions have been used to quantify the variability in crop yield and near-surface soil water (Green and Erskine, 2004) and to identify the scales of variation of soil electrical conductivity (Bekele et al., 2005). Fekete (2001) computed fractal dimension of crop yield by a time series analysis of yield monitor output and the threshing cylinder torque.

The objectives of this study were to introduce an appropriate methodology for fractal analysis of agricultural yield maps and to demonstrate how the developed method can be used to quantify and compare global and local variations in crop yield.

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FRACTAL GEOMETRY

SELF-SIMILARITY AND FRACTALS

The word “fractal” was coined by Benoit B. Mandelbrot in 1975 (Mandelbrot, 1977) and refers to a rough or fragmented geometric shape or a quantity that displays self-similarity on several scales. In other words, a fractal is a geometric shape or a pattern of variation that can be divided into parts, each of which is, at least approximately or statistically, a scaled-down copy of the whole. Mandelbrot (1983) formally defines a fractal as “a set for which the Hausdorff-Besicovitch dimension strictly exceeds the topological di-

dimension.” Therefore, although self-similarity is the fundamental concept behind fractals, not every self-similar object is a fractal. Moreover, as was implied before, exact self-similarity is only seen in mathematical fractals such as the Sierpinski gasket and the Minkowski curve (fig. 1). Fractal patterns that are found in natural objects or phenomena show quasi-self-similarity or statistical self-similarity; in other words, they have numerical or statistical qualities that scale in a power law fashion with the scale of magnification. Simple mathematical fractals are monofractals, that is, they must be scaled by a constant number to achieve self-similarity. Most natural fractals, however, show self-similarity under arbitrary scaling (Chen, 1997). These objects are a mixture of many monofractals and are referred to as multifractals.

FRactal Dimensions

In the computation of the Hausdorff fractal dimension for the Minkowski curve shown in figure 1, the “vel” refers to the volumetric elements used. The fractal is covered by a set of volumetric elements of size r_k . In this study, we have adopted square vels, but any arbitrary shape works as well. The number (N_k) of the vels required to completely cover the object is counted. Then, the size of the vel is reduced, usually in a dyadic form, i.e., $r_{k+1} = r_k/2$, and the new number (N_{k+1}) of vels required to cover the fractal is counted. For mathematical fractals, this step can be iterated indefinitely. But for natural fractals, the power law relationships will saturate at some point because no further detail is seen after a finite number of magnifications (Vicsek, 1992). N_k is assumed to have a power law relationship with r_k :

$$N_k = \left(\frac{1}{r_k} \right)^{D_H} \quad (1)$$

where D_H is called the Hausdorff dimension of the object under consideration (Mandelbrot, 1977). The regression line on a plot of $\log(N_k)$ versus $\log(1/r_k)$ for several vel coverings is the simplest way to find D_H . In other words, in the limit:

$$D_H = \lim_{k \rightarrow \infty} \frac{\log(N_k)}{\log(1/r_k)} \quad (2)$$

For a non-fractal curve, such as a line or a differentiable curve, D_H equals one. For a fractal plane curve, however, it

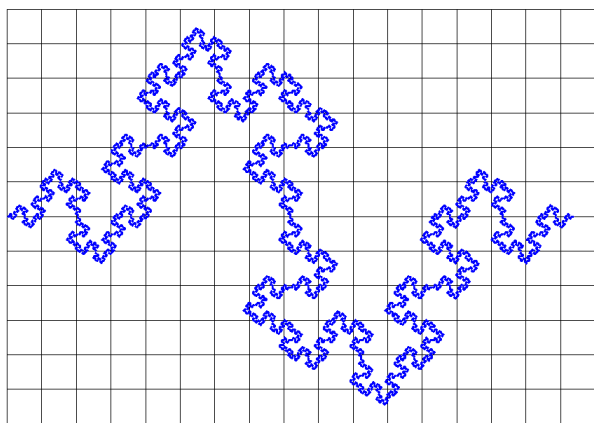


Figure 1. Minkowski curve: An example of mathematical fractals.

will be greater than one and less than or equal to two. For the Minkowski curve shown in figure 1, it equals 1.50.

The Hausdorff dimension belongs to morphology-based fractal dimensions. The value of these fractal dimensions depends only on the morphology of the fractal. More precisely, morphological fractal dimensions quantify the degree of irregularity of the fractal: the higher the degree of irregularity of the fractal, the larger the computed dimension. There are situations in which morphology-based fractal dimensions do not provide a sufficient description of the fractal pattern. For example, in fractal analysis of a strange attractor of a dynamical system, one is interested in time change, or evolution, of the fractal. In fractal analysis of a turbulent flow, one is interested in the distribution of a variable such as fluid pressure or velocity over a spatial fractal. Therefore, a series of entropy-, variance-, and transform-based dimensions have been defined to deal with these applications (Kinsner, 2005). One important variation of entropy-based fractal dimensions is the information dimension D_I (Peitgen et al., 2004):

$$D_I = \lim_{k \rightarrow \infty} \frac{H_{1k}}{\log(1/r_k)} \quad (3)$$

In this equation, H_1 is the Shannon entropy given by:

$$H_{1k} = - \sum_{j=1}^{N_k} p_{jk} \log p_{jk} \quad (4)$$

where p_j is the probability of the fractal entering the j th vel (for strange attractors) or the value of the measure of interest in the j th vel. The value of p_j has to be normalized before being put in equation 4, so that:

$$\sum_{j=1}^{N_k} p_{jk} = 1 \quad (5)$$

The Shannon entropy (H_{jk}) is in bits and quantifies information in a piece of data. Here, it is equivalent to the amount of information necessary to specify a point of the fractal with an accuracy of r_k (Peitgen et al., 2004).

Another important variation of entropy-based dimensions is the correlation dimension (D_C), defined as (Peitgen et al., 2004):

$$D_C = \lim_{k \rightarrow \infty} \frac{-\log \sum_{j=1}^{N_k} p_{jk}^2}{\log(1/r_k)} \quad (6)$$

The numerator in equation 6 is known as the correlation sum and can be interpreted as a correlation between pairs of adjacent points on the fractal (Kinsner, 2005). It is important to mention that equations 2, 3, and 6 assume that a power law relationship exists between the terms that appear in the numerator and the vel size; otherwise, the object or phenomenon under study is not a fractal.

Characterization by a single number, or dimension, is adequate only for those objects that show exact self-similarity (Chen, 1997). In fact, for such objects or phenomena, all of the fractal dimensions result in the same value. As mentioned before, natural fractals are only statistically self-similar and require more than one number to be characterized completely (Kinsner, 2005). The Rényi spectrum of dimension is based on the Rényi generalized entropy (Rényi, 1961), which is a

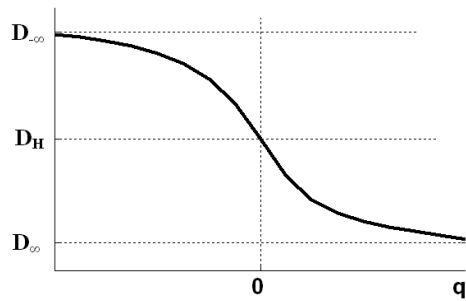


Figure 2. Typical shape of the Rényi spectrum of dimensions for a multifractal object.

generalization of the Shannon entropy (eq. 4) and can be used to characterize the distribution of multifractals. The Rényi spectrum of dimensions is defined as (Peitgen et al., 2004):

$$D_q = \lim_{k \rightarrow \infty} \frac{1}{q-1} \frac{\log \sum_{j=1}^{N_r} p_{jk}^q}{\log(1/r_k)} \quad (7)$$

where q is the moment order, and D_q is the dimension of the multifractal object for a given q . Theoretically, the spectrum can be computed for the whole range of $-\infty < q < \infty$, but in practice, due to limited accuracy in numerical calculations, the spectrum is computed on a limited range, usually $-10 \leq q \leq 10$. Therefore, instead of a single dimension, D_q provides a continuous range, or spectrum, of dimensions. It is easy to show that for $q = 0, 1$, and 2 , the Rényi dimension reduces to Hausdorff, information, and correlation dimensions, respectively (Peitgen et al., 2004). In addition, some other single-valued fractal dimensions correspond to D_q for some value of the moment q ; therefore, the Rényi spectrum can be thought of as the “unifying framework” of all single dimensions (Kinsner, 2005). For monofractals, the value of D_q is constant for all values of q , while for multifractals, it is a monotonically decreasing function of q (fig. 2). Therefore, the Rényi spectrum separates monofractals, for which the spectrum is a horizontal line, from multifractals, for which it is monotonically decreasing (Halsey et al., 1986). This curve can be interpreted as a “fingerprint” of the fractal. A very important feature of this signature is that it has limited bounds; that is, D_∞ and $D_{-\infty}$ are finite.

Perhaps the best way to understand the fractal dimension of a yield variation pattern is to visualize this pattern as a 3D surface. The Hausdorff fractal dimension shows the degree of unevenness of this surface. For a smooth, that is differentiable, surface, $D_H = 2$, while for an extremely rough surface, D_H will increase towards 3. The Rényi spectrum of dimensions can be thought of as a generalization of this concept. Using equation 7, the value of the Rényi dimension (D_q) for large values of q is mostly determined by those vels that have

a high value of the variable under study, whereas for small values of q , D_q is determined mostly by those vels that have a low value of the variable. In other words, for small values of q , D_q indicates roughness of yield variation at low-yield areas, while for large values of q , it shows the same thing mainly for those spots with high yield. This insight is unique to fractal analysis and cannot be obtained using other methods.

METHODS

The study site was a 24 ha field in Ohio, and the period of the study was from 1997 to 2001. The site was under wheat cultivation in 1997 and 1999, under soybean cultivation in 1998 and 2001, and under corn in 2000. A 432×288 m rectangle-shaped central portion of the field was chosen for analysis. This choice was made based on the limitations in terms of the size and shape of the field and the requirement that the size of vel coverings should form a sequence of dyadic numbers.

A grain combine harvester was used for harvesting the corn, wheat, and soybean. During harvesting, the yield data were obtained by a yield monitoring system (AgLeader, Ames, Iowa). The yield monitoring systems were calibrated each season according to the manufacturer’s instructions. The yield monitor software program was able to export the yield data in AgLeader advanced format (Shearer et al., 1999), which is an ASCII format. The spatial resolution of the yield data was determined by the combine head width and the yield log interval. As shown in table 1, the combine head and log interval were 5.9 to 7.6 m and 1.4 to 1.8 m, respectively, so that the spatial resolution was between 10.0 and 15.7 m². The number of data points in the chosen field of 432×288 m was between 7,900 and 12,500. The original data were interpolated to obtain the value of yield on a uniform grid of 2.25 m. Since the map size was 432×288 m, this provided 192 and 128 points in each direction, so that a total of 24,576 data points for the entire map was made. The triangle-based linear interpolation functionality in Matlab (ver. 7.1, The Math Works, Inc., Natick, Mass.) was used for interpolation. Table 1 also shows the mean and coefficient of variation (CV) of the yield data for each of the five years. Although the header width of the combines was larger than 2.25 m, this interpolation did not introduce an error because, as described later, the smallest vel size in this analysis was 4.5 m.

A dithering technique was used to convert the original data points into a set of points in \mathfrak{R}^2 . As mentioned before, the yield was determined by interpolation on a grid of 192×128 points. These data points can be considered as graph of yield versus location in \mathfrak{R}^3 . In the proposed approach, the value of yield at each interpolated point was first mapped to an integer in the range of 1 to 32. This range was chosen based on the reported accuracy for typical grain yield monitors

Table 1. Summary of the yield data.

Crop and Year	Combine Head Width (m)	Yield Log Interval (m)	Resolution of the Data (m ²)	Number of Data Points	Mean Yield (Mg ha ⁻¹)	CV (%)
Wheat in 1997	7.6	1.6	12.2	10,100	4.21	0.133
Soybean in 1998	8.5	1.7	14.5	8,600	3.62	0.144
Wheat in 1999	8.6	1.4	12.0	10,300	4.46	0.130
Corn in 2000	5.9	1.7	10.0	12,500	10.90	0.139
Soybean in 2001	8.7	1.8	15.7	7,900	3.60	0.133

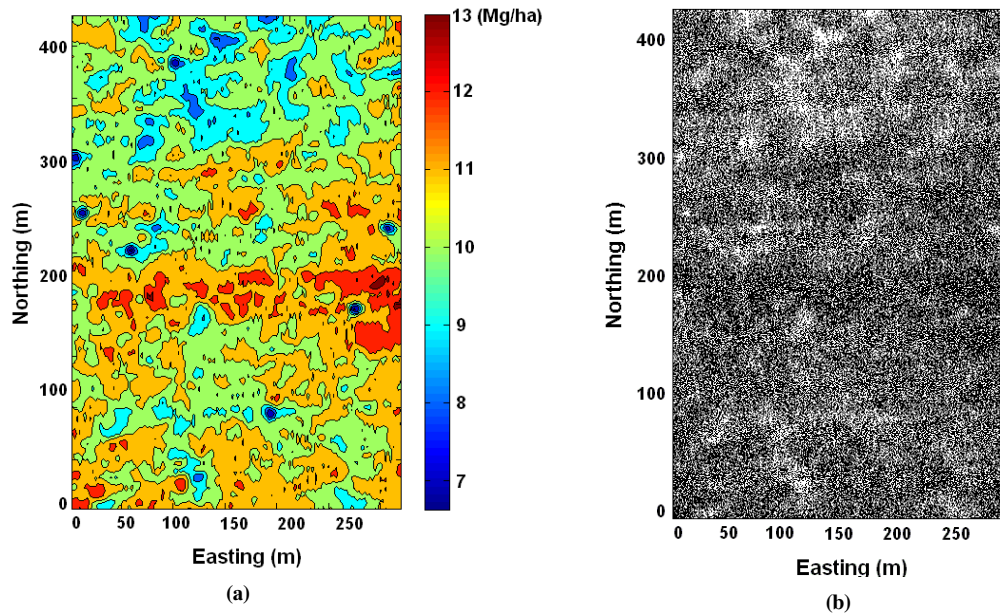


Figure 3. Corn yield map in the year 2000: (a) contour plot, and (b) transformed using the dithering technique.

(Arslan and Colvin, 2002; Loghavi and Ehsani, 2004; Li et al., 2005). Then, each of the points in \mathfrak{R}^3 was replaced by an equivalent number of random points in the corresponding square in the \mathfrak{R}^2 space. For example, if the normalized yield value at a particular location was 20, then 20 random points were placed in the corresponding square of size 2.25 m. An example of such a transformation is shown in figure 3b.

Fractal analysis of the map obtained using this transformation was performed by covering the map with a set of adjacent square vels and computing the Rényi fractal dimension spectrum (eq. 7). In the process of moving from \mathfrak{R}^3 to \mathfrak{R}^2 , no additional information regarding the yield variations is generated. The only reason for this conversion is to increase the reliability of the fractal analysis by moving from a sparse set of points in \mathfrak{R}^3 to a much denser set of points in \mathfrak{R}^2 . Analysis of the dithered fractal in \mathfrak{R}^2 is not the same as the analysis of the original 3D fractal. Analysis of the original fractal would be ideal, but it is not possible due to the small number of points. Dithering provides a much larger number of points in 2D. In other words, the accuracy in measurement of yield is used to convert each point in 3D into a number of points to create a much denser fractal. This is done in order to increase the number of points in the fractal, which is a necessity for a valid analysis.

RESULTS AND DISCUSSION

Figure 4 shows an example of the observed power law relationship. In this figure, the Shannon entropy (H_1) has been plotted against $\log(1/r_k)$ for eight vel coverings for the 1997 wheat yield map. The largest vel size was 48 m, and the smallest one was 0.375 m. The fractal dimension appears as the slope of the regression line. Transforming the data points from \mathfrak{R}^3 into \mathfrak{R}^2 using the dithering technique provided a larger number of points. This allowed choosing vel sizes well below 4.5 m without saturation of the power law relationship.

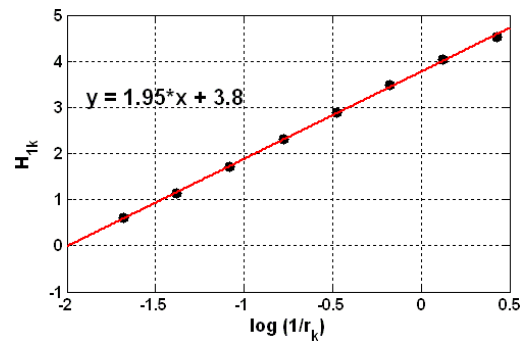


Figure 4. Example of the power law relationship for the 1997 wheat yield map transformed into 2D.

Table 2. Hausdorff dimension (D_H), information dimension (D_I), and correlation dimension (D_C) for the yield maps of different crops and years obtained by an analysis in 2D.

Crop and Year	D_H	D_I	D_C
Wheat in 1997	1.977	1.951	1.930
Soybean in 1998	1.977	1.953	1.934
Wheat in 1999	1.980	1.958	1.939
Corn in 2000	1.985	1.961	1.940
Soybean in 2001	1.985	1.963	1.945

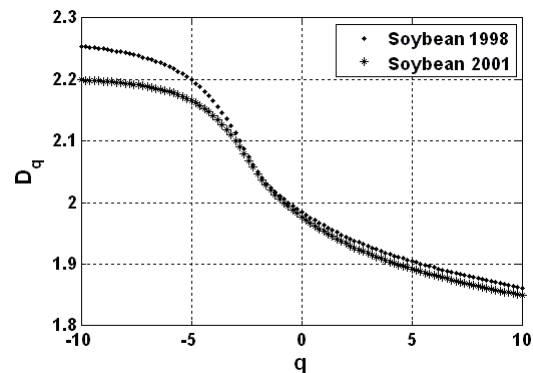


Figure 5. Rényi fractal dimension spectrum for soybean yield in the years 1998 and 2001.

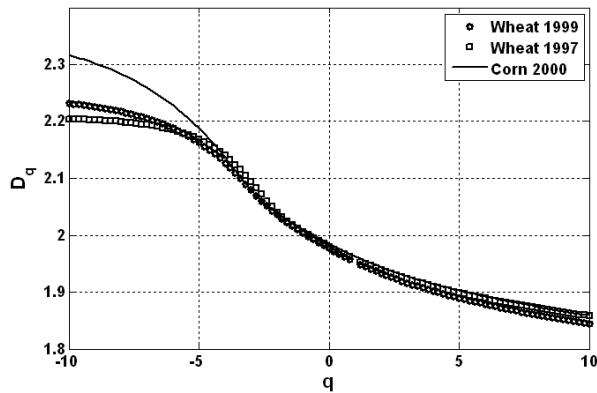


Figure 6. Rényi fractal dimension spectrum for wheat and corn yield maps obtained in the years 1997, 1999, and 2000.

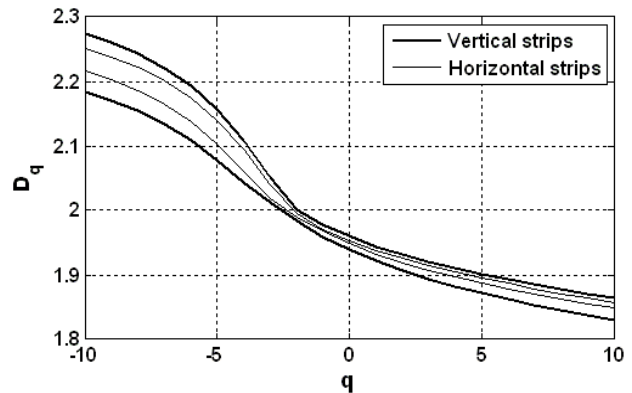


Figure 7. The horizontal and vertical strips that have the largest and smallest Rényi dimensions for the 1999 wheat yield map.

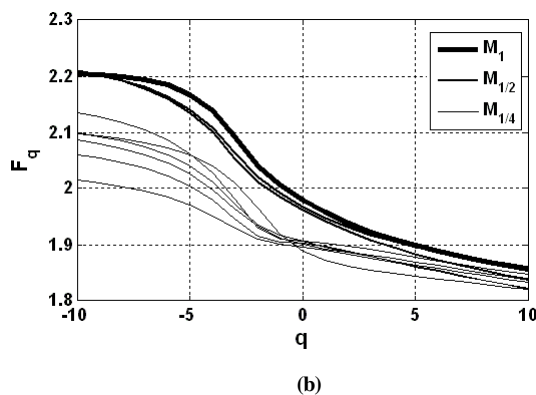
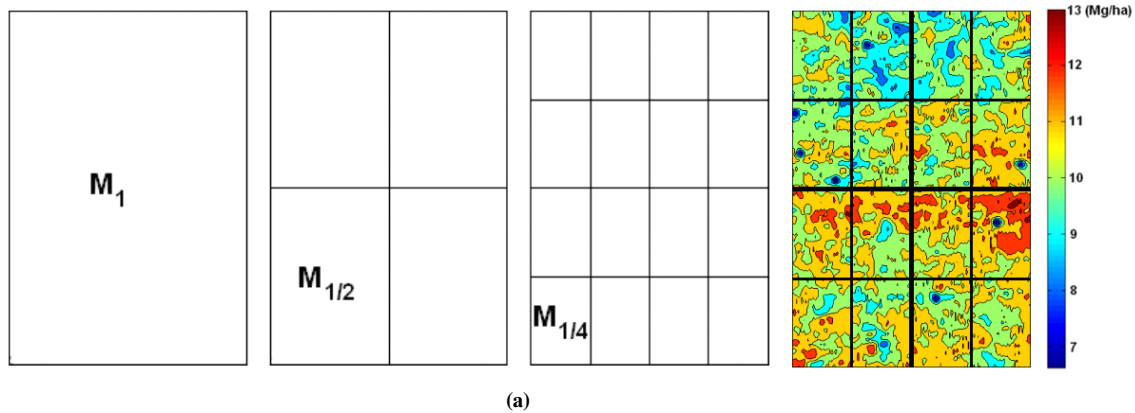


Figure 8. (a) Dyadic segmentation of the yield map and (b) Rényi fractal dimension spectra of M_1 , two $M_{1/2}$, and six $M_{1/4}$ of the 1997 wheat yield map.

Table 2 shows the Hausdorff, information, and correlation dimensions for each of the five years of yield data.

Figure 5 shows the spectrum of dimensions for the years 1998 and 2001 when the field was under soybean cultivation. The most important observation from this figure is that D_q decreases from above 2.2 to below 1.9, which indicates that yield maps are multifractal. From figure 5, it can be said that the variation in soybean yield was very close in 1998 and 2001, but it was slightly higher in 1998. This observation agrees with the fact that the coefficient of variation (CV) of the yield was 0.144 and 0.133 t h⁻¹ in 1998 and 2001, respectively. The Hausdorff dimension (D_q at $q = 0$) is very close to 2 because almost all of the vels contain at least part of the

fractal (i.e., at least one point). It can also be seen that the two graphs are very close at $q = 1$ and 2. Therefore, the information and correlation dimensions are not enough to characterize a specific yield map and the whole spectrum is needed.

The Rényi spectrum of dimensions for the yield maps of the other three years is presented in figure 6. From figure 6, it can be said that for those spots with relatively high yield, degree of yield variation is very close for wheat and corn, but for those areas that produce relatively low yield, corn yield contains more variability than wheat yield. A similar statement can be made about soybean yield variation in 1998 and 2001 (fig. 5); high-yield areas have a nearly equal degree of variation in both years, but low-yield areas have more variation in 1998 than in 2001.

The overall behavior of such complex yield maps does not include all local complexities. In order to study the local behavior of the map, one can segment it into horizontal and vertical strips or into smaller portions of the same shape and study the fractality of these segments (Faghfour et al., 2004). Figure 7 shows the fractal dimension spectrum of the segmented vertical and horizontal strips that had the maximum and minimum Rényi dimension spectra for the 1999 wheat yield map. The original 1999 wheat yield map was first divided into six horizontal strips of height 72 m, and the Rényi fractal dimension spectrum of each strip was computed. Then, the same map was divided into four vertical strips of width 72 m, and the same analysis was completed for each segment. This figure shows that horizontal strips are more alike than vertical strips in terms of smoothness or roughness of their yield variation pattern.

Alternatively, the entire map can be segmented simultaneously in horizontal and vertical directions, as shown in figure 8a, and the Rényi spectra for the subregions can be computed. Figure 8b shows the result of such an analysis for the subregions of the wheat yield map of the year 1997.

In figure 8b, M_1 shows the Rényi dimension of the entire map, and two $M_{1/2}$ and six $M_{1/4}$ represent typical Rényi dimension spectra for subregions of the map. Figure 8b shows that the degree of variation in $M_{1/2}$ subregions is very close to the variation in the entire map. However, $M_{1/4}$ subregions have smaller dimensions as well as smaller variation in their dimension over the range of $-10 \leq q \leq 10$. Figure 8b also shows that while the dimension spectra of $M_{1/2}$ segments are very close, the dimension spectra of $M_{1/4}$ segments vary more significantly. This kind of analysis can be an effective tool for finding appropriate management zones. Figure 8b shows that in this case there is not much difference between $M_{1/2}$ zones, while $M_{1/4}$ zones are quite different in terms of yield variability. Those $M_{1/4}$ zones that have larger fractal dimension contain more variability compared to those with smaller dimension. Therefore, if the field is to be divided into several management zones, then zones of size $M_{1/4}$ would be appropriate, and the Rényi dimension spectra can be used to compare the degree of yield variability in different zones.

This research was aimed to suggest a new approach based on the concepts of self-similarity and fractals to analyze yield variation in crop yield maps. In the last two decades, studies have shown that many random-appearing phenomena in nature are dominated by the rules of chaos theory (Peitgen et al., 2004). These chaotic processes will always produce self-similar patterns. It is likely that the major processes that result in yield variability are also chaotic. If this is true, then fractal analysis is in fact the natural approach for analyzing yield variations. An investigation of this hypothesis, however, was not the objective of this study. As the next phase of this study, the characteristics of the fractal analysis and geostatistical method for analyzing multiple years of yield data will be investigated and compared.

CONCLUSIONS

The goal of this study was to investigate fractality of agricultural yield maps. A dithering technique was proposed to transform the original data points into a 2D map. This approach produces a dense set of points and will result in reliable computations. Computation of the Rényi fractal dimension spectrum established the multifractality of yield maps, the major finding of our study. It was shown, by some examples, that this method can be used to analyze and compare yield spatial variability. Studying the change in the power law relationship with scale can be quite informative. A stronger or weaker power law relationship, i.e., a power law relationship with a larger or smaller exponent, on some scale would indicate rougher or smoother variations on that scale. An illuminating feature of the Rényi spectrum is that it gives different weights to low- and high-yield areas based on the value of the moment q . The results indicate that roughness of yield variations for areas with relatively high yield is very close for all five years of data and therefore might be attributable to the characteristics of the terrain, while for low-yield spots, the differences are more pronounced.

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